MATH 3235 Probability Theory

9/27/22

X Takes only positive integer values.

Probability generating function

pgf.

$$G_X(s) = \sum_{x=0}^{\infty} p(x)\, s^x$$

$$p(x) = \mathbb{P}(X = x)$$

Binomial p N Takes only values $0 \dots N$. ($p(x) = 0$ if $x > N$).

$$G_X(s) = \mathbb{E}(s^X)$$

Since by def $0 \leq p(x) \leq 1$ Then

$G_X(s)$ exists for $|s| < 1$

Uniqueness: if X and Y are integer valued r.v. and

$$G_X(s) = G_Y(s)$$

Then

$$\mathbb{P}(X = k) = \mathbb{P}(Y = k) \quad \forall k$$

X is Bernoulli r.v.

$$\mathbb{E}(s^X) = q + ps = G_X(s)$$

$$\mathbb{P}(X = 0) = G_X(0) = q$$

$$\mathbb{P}(X = 1) = G_X'(0) = p$$

$$\mathbb{P}(X = k) = \frac{1}{k!} G_X^{(k)}(0) = 0 \quad \text{if } k > 1.$$

Take a binomical p N

$$p(x) = \binom{N}{x} p^x (1-p)^{N-x}$$

$$G_X(s) = \sum_{x=0}^{N} s^x p^x (1-p)^{N-x} \binom{N}{x} =$$

$$\sum_{x=0}^{N} (sp)^x (1-p)^{N-x} \binom{N}{x} = (q+sp)^N$$

Thm: if X and Y are integer valued and $X \perp\!\!\!\perp Y$

Then

$$G_{X+Y}(s) = G_X(s) \, G_Y(s)$$

$$P_{X+Y}(z) = \sum_x P_Y(z-x) \, P_X(x)$$

X is binomial with par

N, p

$$X = \sum_{i=1}^{N} Y_i$$ where Y_i are i.i.d.

Bernoulli par p.

$$G_{Y_i}(s) = q + ps$$

$$G_X(s) = (q+ps)^N = \left(G_{Y_i}(s)\right)^N$$

N is Poisson par λ.

$$p(n) = \frac{\lambda^n}{n!} e^{-\lambda}$$

$$G_N(s) = \sum_{n=0}^{\infty} s^n \frac{\lambda^n}{n!} e^{-\lambda} = e^{s\lambda} e^{-\lambda} =$$

$$= e^{\lambda(s-1)}$$

N Poisson and

X_i are i.i.d. Bernoulli p

$$M = \sum_{i=1}^{N} X_i$$

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X and Y

$$G_{X+Y}(s) = \mathbb{E}\left(s^{X+Y}\right) =$$

$$= \mathbb{E}\left(s^X s^Y\right) =$$

$$\boxed{\mathbb{E}\left(f(X) g(Y)\right) \overset{X \perp\!\!\!\perp Y}{=} \mathbb{E}\left(f(X)\right)\mathbb{E}\left(g(Y)\right)}$$

$$G_{X+Y}(s) = \mathbb{E}\left(s^X\right)\mathbb{E}\left(s^Y\right) = G_X(s)\, G_Y(s)$$

$$\mathbb{E}\left(s^{\sum_{i=1}^{N} X_i}\right) =$$

$$\mathbb{E}\left(s^{\sum_{i=1}^{n} X_i} \,\middle|\, N = n\right) \mathbb{P}(N = n)$$

X_i are i.i.d.

N, X_i is a family of indep.

r.v.

$$G_M(s) = \mathbb{E}\left(s^{\sum_{i=0}^{N} X_i}\right) =$$

$$= \sum_n \mathbb{E}\left(s^{\sum_{i=0}^{n} X_i} \,\middle|\, N = n\right) \mathbb{P}(N = n) =$$

$$= \sum_n \left(G_X(s)\right)^n \mathbb{P}(N = n) =$$

$$G_N(s) = \sum_n s^n \mathbb{P}(N = n)$$

$$G_M(s) = G_N(G_X(s))$$

Thm:

if X_i are i.i.d.

N is ind. from X_i

$$M = \sum_{i=0}^{N} X_i$$

Then

$$G_M(s) = G_N(G_X(s))$$

$G_N(s) = e^{\lambda(s-1)}$ N Poiss.

$G_X(s) = q + ps$

$$G_M(s) = e^{\lambda(q+ps-1)} = e^{\lambda p(s-1)}$$

$q = 1 - p$

M is Poissonian with

par λp.

Moments.

$$\mathbb{E}(X^n) = \text{n-th moment}$$

$$m_1 = \mathbb{E}(X)$$

$$m_2 = \mathbb{E}(X^2)$$

$$\operatorname{var}(X) = m_2 - m_1^2$$

$$\frac{d}{ds}\mathbb{E}(s^X) = \mathbb{E}(X s^{X-1})$$

$$\mathbb{E}(X^2) - \mathbb{E}(X)^2 > 0$$

$$\shortparallel$$

$$\mathbb{E}((X - \mathbb{E}(X))^2)$$

$$\frac{d}{ds}\mathbb{E}(s^X) = \mathbb{E}(X s^{X-1})$$

$$\frac{d}{ds} G_X(1) = \mathbb{E}(X)$$

$$G_X''(s) = \mathbb{E}\left(X(X-1)\, s^{X-2}\right) =$$

$$G_X''(1) = \mathbb{E}\left(X(X-1)\right) =$$
$$= \mathbb{E}(X^2) - \mathbb{E}(X)$$

$$\operatorname{var}(X) = G_X''(1) + G_X'(1) - \left(G_X'(1)\right)^2$$

$$G_X'''(1) = \mathbb{E}\left(X(X-1)(X-2)\right)$$

Distribution function.

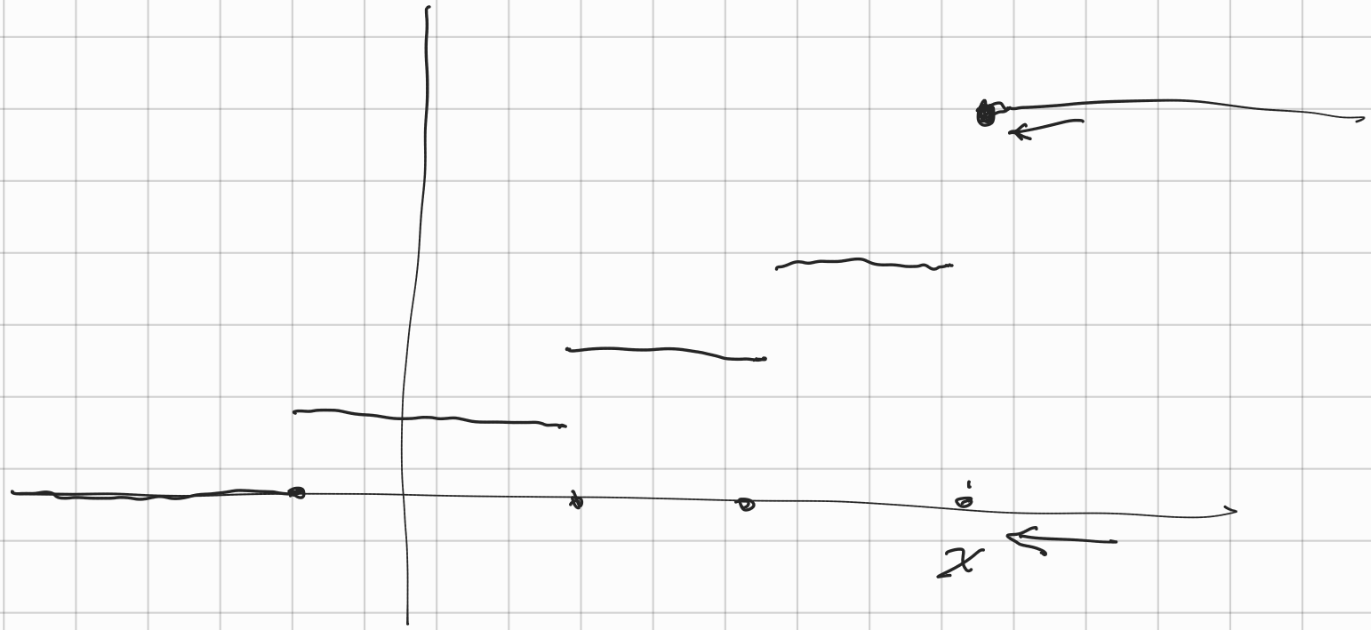
X is a r.v.

$F(x) = \mathbb{P}(X \leq x)$ for every x

1) $F(x)$ is defined for every x

2) $F(x)$ not decreasing in x

3) $F(-\infty) = 0$ $F(\infty) = 1$



$$\lim_{y \to x^+} F(y) = F(x)$$